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Remark on uniqueness of mild solutions to the Navier–Stokes equations

Hideyuki Miura

Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

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Abstract

We investigate a limiting uniqueness criterion to the Navier–Stokes equations. We prove that the mild solution is unique under the class $C([0,T); bmo^{-1}) \cap L^\infty_{\mathrm{loc}}((0,T); L^\infty)$, where bmo^{-1} is the "critical" space including L^n. As an application of uniqueness theorem, we also consider the local well-posedness of Navier–Stokes equations in bmo^{-1}.

Keywords: Navier–Stokes; Uniqueness; Mild solution; bmo^{-1}

1. Introduction

Let us consider the Navier–Stokes equations in $\mathbb{R}^n$:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\ u|_{t=0} = a \quad \text{in } \mathbb{R}^n, \end{cases} \tag{NS}$$

where $u = u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x,t) \in \mathbb{R}^n \times (0,T)$, respectively, while $a = a(x) = (a_1(x), \ldots, a_n(x))$ is the given initial velocity vector.

E-mail address: sa1m25@math.tohoku.ac.jp (H. Miura).

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There are a number of results on existence and uniqueness of solutions of (NS) [5,10]. Kato [9] and Giga–Miyakawa [8] proved that for $a \in L^n(\mathbb{R}^n)$ there exist $T > 0$ and at least one solution u in

$$C([0,T); L^n) \cap C((0,T); L^p) \qquad (n < p \leqslant \infty) \tag{CL_T^p}$$

such that u solves (NS) in the sense of the following integral equation:

$$u(t) = e^{t\Delta}a - \int_0^t e^{(t-s)\Delta}\mathrm{P}\nabla \cdot (u \otimes u)(s)\,ds, \tag{NSI}$$

where $e^{t\Delta} = G_t *$ denotes the heat semigroup and P denotes the Helmholtz–Weyl projection. Such u is called a mild solution of (NS).

As for uniqueness, the authors showed that under the additional condition

$$\lim_{t\to 0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}\|u(t)\|_{L^p} = 0 \qquad (n < p \leqslant \infty), \tag{Ad}$$

the mild solution is unique in CL_T^p. This condition is regarded as restriction of behavior on L^p norm of solutions in the neighborhood at $t = 0$.

Brezis [3] proved that every mild solution in CL_T^p necessarily satisfies (Ad), so he clarified that (Ad) is, in fact, redundant for uniqueness of mild solutions.

The purpose of the present paper is to generalize the criterion of uniqueness. We shall prove that the mild solution u in $L^2(0,T; L^2_{uloc})$ is unique under the class

$$C([0,T); bmo^{-1}) \cap L^\infty_{\mathrm{loc}}((0,T); L^\infty)$$

for the initial value in vmo^{-1}. bmo^{-1} coincides the Triebel–Lizorkin space $F^{-1}_{\infty,2}$, which contains L^p functions $(n \leqslant p \leqslant \infty)$ and derivatives of BMO functions. vmo^{-1} is a subspace in bmo^{-1}. Since we can replace $L^\infty_{\mathrm{loc}}((0,T); L^\infty)$ by $L^s_{\mathrm{loc}}((0,T); L^p)$ $(2/s + n/p = 1, n < p < \infty)$ and since both bmo^{-1} and vmo^{-1} include L^n, this class is larger than earlier classes such as CL_T^p. Particularly it should be noted that we can replace $C([0,T); L^n)$ by $C([0,T); bmo^{-1})$. Behavior of the solution near $t = 0$ plays an essential role for validity of uniqueness of mild solutions. Indeed, on account of the smoothing effect, its behavior away from $t = 0$ less contributes to uniqueness.

Our result is inspired by Koch–Tataru's existence theorem [10]. They proved the (local) existence of mild solutions, when $a \in vmo^{-1}$ (In [10] vmo^{-1} denotes $\overline{VMO^{-1}}$. We use the notations of the paper of Bourdaud–Lanza de Cristoforis–Sickel [2]). It extended the class of the initial value for which the mild solution exists. They also mentioned the relation between the initial value and the existence time of the mild solution by introducing the BMO_T^{-1} norm. It was proved that if the BMO_T^{-1} norm of the initial value is sufficiently small, there exists at least one mild solution on $[0, T)$. Here we notice the fact that every vmo^{-1} function f satisfies $\lim_{T\to 0}\|f\|_{BMO_T^{-1}} = 0$.

In the proof of our theorem, it is essential to show that the (Ad)-type condition can be obtained necessarily from our assumption. In [3], the author noticed that the subset $\{u(\tau); 0 < \tau < T\}$ of solution in CL_T^p is a precompact subset in L^n. Making use of the fact that local existence time-interval can be taken uniformly in each precompact subset of initial values, he identified every mild solution in CL_T^p as the special solution with (Ad) which can be constructed by usual iteration procedure. On the other hand, in order to obtain (Ad)-type condition, we first establish the following fact; if the mild solution u exists on $[0, T)$ with the initial value $a = u(0)$, then for any time τ near $t = 0$, we can construct another mild solution $\tilde{u}$ with $\tilde{u}|_{t=0} = u(\tau)$ having a better property than the original u. To this end, it plays an important role to estimate the BMO_T^{-1} norm of $u(\tau)$. The advantage of our method is that we do not need any density of C_0^∞ for the space where uniqueness is discussed, and that it rather simplify the proof. See also Theorem 2.3 Remarks.

We turn to the problem of the local well-posedness in vmo^{-1}. For our uniqueness criterion, we need continuity of the solution $u(t)$ in $t \in [0, T)$ as a bmo^{-1}-valued function. Although Koch-Tataru constructed a mild solution for the initial value in vmo^{-1}, it seems to be unknown, in general, whether their solution belongs to $C([0, T); bmo^{-1})$. The lack of continuity seems to stem from the fact that vmo^{-1} is too large for the operator $e^{t\Delta}$ to become a "strongly continuous" semigroup. To get around such difficulty, we introduce a new class gmo^{-1} of the initial value, and show that there exists a "unique" solution in $C([0, T); bmo^{-1})$ for some $T < \infty$. Our space gmo^{-1} is slightly smaller than bmo^{-1}. However, we obtain local well-posedness, i.e., existence of local solutions, its uniqueness and continuity in time.

2. Definitions and statements of theorems

Before stating our result, we introduce some function spaces.

Definition 2.1. (i) Suppose that f is a measurable function in $\mathbb{R}^n$. We write

$$\|f\|_{BMO_T^{-1}} := \sup_{x\in\mathbb{R}^n,\ 0<R^2<T} \left(\frac{1}{|B(x, R)|} \int_{B(x,R)} \int_0^{R^2} |e^{t\Delta} f(y)|^2 \, dt\, dy \right)^{\frac{1}{2}},$$

where $e^{t\Delta} = G_t*$ denotes the heat semigroup and $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ denotes the heat kernel. Then we define function spaces as follows:

$$BMO^{-1} := \left\{ f \in \mathcal{S}'(\mathbb{R}^n);\ \|f\|_{BMO^{-1}} := \|f\|_{BMO_\infty^{-1}} < \infty \right\},$$

$$bmo^{-1} := \left\{ f \in \mathcal{S}'(\mathbb{R}^n);\ \|f\|_{bmo^{-1}} := \|f\|_{BMO_1^{-1}} < \infty \right\},$$

$$vmo^{-1} := \left\{ f \in bmo^{-1};\ \lim_{T\to 0} \|f\|_{BMO_T^{-1}} = 0 \right\},$$

$$gmo^{-1} := \left\{ f \in bmo^{-1};\ \lim_{t\to 0} e^{t\Delta} f = f \quad \text{in } bmo^{-1} \right\}.$$

(ii) Suppose that u is a measurable function in $\mathbb{R}^n \times [0, T)$. We write

$$\|u\|_{\mathcal{E}_T} := \sup_{0<t<T} t^{\frac{1}{2}} \|u(t)\|_{L^\infty} + \sup_{x\in\mathbb{R}^n,\ 0<R^2<T} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} \int_0^{R^2} |u(y,t)|^2 \, dt \, dy \right)^{\frac{1}{2}}.$$

Then the space $\mathcal{E}_T$ is defined by

$$\mathcal{E}_T := \left\{ u \in L^2(0, T; L^2_{\mathrm{loc}});\ \|u\|_{\mathcal{E}_T} < \infty \right\}.$$

Remarks. (1) The spaces in (i) are considered as the spaces of the initial data to (NS). On the other hand, $\mathcal{E}_T$ is the space in which we find a solution of the evolution equation.

(2) BMO^{-1} consists of the first derivatives of functions in *BMO*. It is well known that $\log|x| \in BMO$, so a typical function in BMO^{-1} is $|x|^{-1}$. BMO^{-1} includes the scaling invariant-spaces such as L^n.

(3) bmo^{-1} consists of the sum of functions in *bmo* and its first derivatives, where $bmo = BMO \cap L^1_{uloc}$. In particular bmo^{-1} also includes BMO^{-1}. vmo^{-1} and gmo^{-1} are closed subspaces of bmo^{-1}. They contain L^p $(n \leqslant p \leqslant \infty)$.

(4) The $\mathcal{E}_T$ norm is related to that of BMO_T^{-1} via the heat semigroup. Indeed for the solution of heat equation $u_0(t) = e^{t\Delta}a$, it holds that $\|u_0\|_{\mathcal{E}_T} \simeq \|a\|_{BMO_T^{-1}}$. Koch and Tataru [10] showed that there exists constant ε_0 such that if $\|a\|_{BMO_T^{-1}} < \varepsilon_0$, there exists a mild solution in $\mathcal{E}_T$. In particular, they proved that for $a \in vmo^{-1}$ there exist $T > 0$ and a mild solution of (NS) in the class $\mathcal{E}_T$. As for Koch–Tataru's result, see also [1,7,12,14].

Next we define a notion of the mild solution.

Definition 2.2. Let $a \in \mathcal{S}'$. A measurable function u is called a (uniformly locally square integrable) mild solution of (NS) on $(0, T)$, if u belongs to $L^2(0, T; L^2_{uloc})$ and if u satisfies

$$u(t) = e^{t\Delta}a - B(u, u)(t) \qquad \text{on } (0, T), \tag{NSI}$$

where

$$L^2(0,T;L^2_{uloc}) := \left\{ u \in L^2(0,T;L^2_{\rm loc});\ \int_0^T \sup_{x\in\mathbb{R}^n} \int_{B(x,1)} |u(y,s)|^2\,dy\,ds < \infty \right\},$$

$$B(u,v)(t) := \int_0^t e^{(t-s)\Delta}\mathrm{P}\nabla\cdot(u\otimes v)(s)\,ds.$$

Here P is the Helmholtz–Weyl projection. More precisely, $\mathrm{P} = \{P_{ij}\}_{i,j=1,\dots,n}$ is represented as $P_{ij} = \delta_{ij} + R_i R_j$, where δ_{ij} is the Kronecker symbol and $R_i = \frac{\partial}{\partial x_i}(-\Delta)^{-\frac{1}{2}}$ are Riesz transforms. If u belongs to $L^2(0,T;L^2_{uloc})$, the right-hand side of (NSI) is well-defined as the $L^1(0,T;L^1_{uloc})$ function. For detail, see Lemarié–Rieusset [12, Chapter 11].

Now we state our uniqueness result.

Theorem 2.3. *Assume that u and v are mild solutions for the same initial value $a \in vmo^{-1}$. If both u and v belong to*

$$C([0,T);bmo^{-1}) \cap L^\infty_{\rm loc}((0,T);L^\infty),$$

then

$$u \equiv v \quad on\ [0,T).$$

Remarks. (1) We can also prove the uniqueness replacing $L^\infty_{\rm loc}((0,T);L^\infty)$ by $L^s_{\rm loc}((0,T);L^p)$ $(2/s+n/p=1,\ n<s<\infty)$. Indeed it is not difficult to see the uniqueness of mild solutions in $L^s(0,T;L^p)$. Hence our assumption implies the uniqueness on $[\varepsilon,T)$ for arbitrary $0<\varepsilon<T$ and we can easily obtain the result by arranging the proof of Theorem 2.3.

(2) Chemin [4] proved the uniqueness of weak solutions in the class

$$C([0,T);B^\infty) \cap L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$$

for the initial value $B^p \cap L^2$ $(p<\infty)$, where B^p denotes the closure of C_0^∞ in the Besov norm of $B^{-1+n/p}_{p,\infty}$. Although it holds that $B^p \hookrightarrow vmo^{-1}$ $(p<\infty)$ and $bmo^{-1} \hookrightarrow B^{-1}_{\infty,\infty}$, there seems not to be simple relations between this result and that of ours. Notice that C_0^∞ is not dense in bmo^{-1} unlike B^p. This fact plays an important role to Theorem 2.4.

(3) The assumption $u \in L^\infty_{\rm loc}((0,T);L^\infty)$ is not unnatural, since

$$\mathcal{E}_T \subset L^2(0,T;L^2_{uloc}) \cap L^\infty_{\rm loc}((0,T);L^\infty).$$

In [6,13,15] for uniqueness, they need only to assume that the mild solution u is in $C([0,T);L^n)$. One of the reason is that (NSI) is well-defined only if u belongs to $C([0,T);L^n)$ ($\subset L^2(0,T;L^2_{uloc})$). Since the norm bmo^{-1} is much weaker than that of L^n, it is not clear that the nonlinear term $B(u,u)$ is well-defined for $u \in C([0,T);bmo^{-1})$.

(4) Except for the assumption that $u \in C([0,T);bmo^{-1})$, our result does not require any continuity in time such as $C((0,T);L^p)$ in CL^p_T. Instead of continuity, we make fully use of lower semi-continuity of the supremum norm.

Next we consider the local well-posedness in vmo^{-1}. In [10], the authors proved the existence of mild solutions for the initial value in vmo^{-1}. However they did not mention about smoothing effect, or continuity in time with its value in the Banach space. On the other hand, our uniqueness criterion needs the "extra" assumption that u belongs to $C([0,T);bmo^{-1})$. In order to fulfill this gap, we use the suitable class for the initial value gmo^{-1}, and we obtain the following result.

Theorem 2.4. *Assume that the initial value $a \in vmo^{-1} \cap gmo^{-1}$. Then there exist $T > 0$ and a unique mild solution in*

$$C([0,T);gmo^{-1}) \cap \mathcal{E}_T.$$

Remarks. (1) Koch–Tataru [10] constructed a mild solution for the initial value in vmo^{-1}. Theorem 2.4 shows that under the additional condition gmo^{-1} for the initial value, the mild solution constructed by Koch–Tataru possesses time-continuity in gmo^{-1}. Since $C([0,T);\ gmo^{-1}) \cap \mathcal{E}_T$ is contained by our uniqueness class, Theorem 2.3 indicates the uniqueness of the mild solution.

(2) Although both vmo^{-1} and gmo^{-1} are closed subspaces of bmo^{-1}, the relation between two spaces is not clear. It seems to be an interesting problem whether vmo^{-1} is strictly larger than the class $vmo^{-1} \cap gmo^{-1}$. If there is the initial value $a \in vmo^{-1}$ which does not belong to gmo^{-1}, we can prove the existence of the mild solution which is not continuous in bmo^{-1} since we observe that the nonlinear term is still continuous in bmo^{-1} by the proof of Theorem 2.4. We will discuss the relation between vmo^{-1} and gmo^{-1} in the forthcoming paper.

3. Proof of theorems

3.1. Proof of Theorem 2.3

First, we shall prove Theorem 2.3 under the additional assumption (Ad$'$) which is similar to (Ad). Next we shall show that (Ad$'$) is, in fact, redundant.

In the proof, it plays an important role to investigate behavior of the nonlinear term $B(u,u)$ in $\mathcal{E}_T$. For that purpose, let us recall the following bilinear estimate obtained by Koch–Tataru.

Lemma 3.1 (*Koch–Tataru [10]*). (i) *Let the space* $\mathcal{E}_T$ *and the nonlinear term* $B(u,v)$ *be as in Definitions* 2.1, *and* 2.2, *respectively*. *There exists a constant* $c_2 > 0$ *such that*

$$\|B(u,v)\|_{\mathcal{E}_T} \leqslant c_2 \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \quad \textit{for } T>0 \textit{ and } u,v \in \mathcal{E}_T. \tag{3.1}$$

(ii) *There exists a constant* c_1 *such that if* $\|a\|_{BMO_T^{-1}} < 1/(4c_1c_2) \equiv \varepsilon_0$, *there exists a mild solution* u *in the class* $\mathcal{E}_T$ *such that*

$$\|u\|_{\mathcal{E}_T} \leqslant c_1 \|a\|_{BMO_T^{-1}} + c_2 \|u\|_{\mathcal{E}_T}^2, \tag{3.2}$$

where c_1 *and* c_2 *are constants independent of* $T>0$ *and* u, v.

We see the first step by using the estimate (3.1).

Proposition 3.2. *Let* u *and* v *be mild solutions in* $L^\infty_{\mathrm{loc}}((0,T);L^\infty)$ *with the same initial value* $a \in \mathcal{S}'$. *Assume that* u *and* v *satisfy*

$$\lim_{t\to 0} \|u\|_{\mathcal{E}_t} = 0 \quad \lim_{t\to 0} \|v\|_{\mathcal{E}_t} = 0. \tag{Ad$'$}$$

Then we have

$$u(t) \equiv v(t) \qquad \textit{on } [0,T).$$

Proof. Put $w := u - v$. It follows from (3.1) that

$$\begin{aligned}
\|w\|_{\mathcal{E}_t} &= \|B(u,u) - B(v,v)\|_{\mathcal{E}_t} \\
&= \|B(u+v, u-v)\|_{\mathcal{E}_t} \\
&\leqslant c_2 \|u+v\|_{\mathcal{E}_t} \|u-v\|_{\mathcal{E}_t} \\
&\leqslant c_2 (\|u\|_{\mathcal{E}_t} + \|v\|_{\mathcal{E}_t}) \|w\|_{\mathcal{E}_t}.
\end{aligned}$$

By the assumption (Ad$'$), there exists $t_0 > 0$ such that

$$\|u\|_{\mathcal{E}_{t_0}} + \|v\|_{\mathcal{E}_{t_0}} \leqslant \frac{1}{2c_2}.$$

Hence we obtain

$$\|w\|_{\mathcal{E}_{t_0}} \leqslant \frac{1}{2}\|w\|_{\mathcal{E}_{t_0}},$$

from which it follows that

$$u \equiv v \quad \text{on } [0, t_0).$$

We shall next extend the uniqueness to $[0, T)$. Since u belongs to $L^\infty_{\mathrm{loc}}((0, T);\ L^\infty)$, we have

$$\sup_{t_0<s<T} \|u(s)\|_{L^\infty} + \sup_{t_0<s<T} \|v(s)\|_{L^\infty} \equiv M < \infty.$$

To complete the proof of Proposition 3.2, we may show the following lemma.

Lemma 3.3. *There is a constant* $\xi = \xi(n, t_0, T)$ *such that if* $u(t) \equiv v(t)$ *on* $[0, \delta)$ *for some* δ *in* $[t_0, T)$*, then* $u(t) \equiv v(t)$ *holds on* $[0, \delta + \xi)$.

Proof. It suffices to show that there is $\xi = \xi(n, t_0, T)$ such that

$$D(\delta + \xi) \leqslant \frac{1}{2} D(\delta + \xi),$$

where $D(\tau) := \sup_{\delta<s<\tau} \|w(s)\|_{L^\infty}$.

Since this estimate is translation invariant in the space variable, we may see the following estimate:

$$|w(0, t)| \leqslant \frac{1}{2} D(\delta + \xi) \quad \text{for } t \in [\delta, \delta + \xi). \tag{3.3}$$

To this end, we regard the nonlinear term B as a bilinear integral operator with the expression

$$B(u, v)(x, t) := \int_{\mathbb{R}^n} \int_0^t K(x - y, t - s)(u \otimes v)(y, s)\, ds\, dy,$$

where $K(x, t) = \nabla \mathrm{P} G_t(x)$. Notice that

$$|K(x, t)| \leqslant c(t^{\frac{1}{2}} + |x|)^{-n-1} \quad \text{for } t > 0,\ x \in \mathbb{R}^n. \tag{3.4}$$

For the proof, see e.g. Lemarié–Rieusset [12, Chapter 11].

There holds

$$
\begin{aligned}
|w(0,t)| &\leqslant \left| \int_{\mathbb{R}^n} \int_{\delta}^{t} K(0-y, t-s)((u \otimes u) - (v \otimes v))(y,s)\, ds\, dy \right| \\
&\leqslant c \int_{\mathbb{R}^n} \int_{\delta}^{t} ((t-s)^{\frac{1}{2}} + |y|)^{-n-1} |(u^2 - v^2)(y,s)|\, ds\, dy \\
&\leqslant c \,(\sup_{\delta < s < t} \|(u^2 - v^2)(s)\|_{L^\infty}) \int_{\mathbb{R}^n} \int_{\delta}^{t} ((t-s)^{\frac{1}{2}} + |y|)^{-n-1}\, ds\, dy \\
&\leqslant c\, M D(t) \int_{\mathbb{R}^n} \int_{\delta}^{t} ((t-s)^{\frac{1}{2}} + |y|)^{-n-1}\, ds\, dy \\
&\leqslant c\, M D(t) \int_{\mathbb{R}^n} \int_{\delta}^{t} (t-s)^{-\frac{1}{2}} (1 + |y'|)^{-n-1}\, ds\, dy' \\
&\leqslant c\, M D(t) \Big[-(t-s)^{\frac{1}{2}} \Big]_{\delta}^{t} \\
&\leqslant c\, M D(t) (t-\delta)^{\frac{1}{2}}.
\end{aligned}
$$

Taking ξ as

$$
\xi := \frac{1}{2cM} + t_0,
$$

we obtain (3.3) since $D(t)(t-\delta)^{\frac{1}{2}}$ is monotone increasing for t $(> \delta)$. $\square$

Next we shall show that the condition (Ad′) is, in fact, redundant for uniqueness.

Proposition 3.4. *Let $a \in vmo^{-1}$. Every mild solution u in the class $C([0,T);\ bmo^{-1}) \cap L^\infty_{\mathrm{loc}}((0,T); L^\infty)$ fulfills the condition* (Ad′)*, that is*

$$
\lim_{t \to 0} \|u\|_{\mathcal{E}_t} = 0.
$$

For the proof of Proposition 3.4, the following lemma plays an important role to prove Proposition 3.4.

Lemma 3.5. *Let $a \in vmo^{-1}$ and let $u \in C([0,T); bmo^{-1})$. For any $\varepsilon > 0$, there exist $\tau_0 > 0$ and $T' > 0$ such that for every $\tau \in (0, \tau_0)$, we can construct a mild solution u_τ with the following property*:

$$u_\tau \in \mathcal{E}_{T'}, \tag{3.5}$$

$$u_\tau(x,0) = u(x,\tau), \tag{3.6}$$

$$\sup_{0<\tau<\tau_0} \|u_\tau\|_{\mathcal{E}_t} < \varepsilon \qquad \textit{for } 0 < t < T'. \tag{3.7}$$

Proof of Lemma 3.5. Without loss of generality, we may assume $\varepsilon < 1/(2c_2)$, where c_2 is the same constant as in Lemma 3.1.

Since $u(\tau)$ is continuous in bmo^{-1} at $\tau = 0$, there exists $\tau_0 > 0$ such that

$$\sup_{0<\tau<\tau_0} \|u(\tau) - a\|_{BMO_1^{-1}} = \sup_{0<\tau<\tau_0} \|u(\tau) - a\|_{bmo^{-1}} < \frac{\varepsilon}{4c_1},$$

where c_1 is the constant in Lemma 3.1.

On the other hand, since a belongs to vmo^{-1}, there exists $T' \in (0,1]$ such that

$$\|a\|_{BMO_t^{-1}} < \frac{\varepsilon}{4c_1} \quad \text{for } 0 < t < T'.$$

Hence we have

$$\begin{aligned}
\sup_{0<\tau<\tau_0} \|u(\tau)\|_{BMO_t^{-1}} &\leqslant \sup_{0<\tau<\tau_0} \|u(\tau) - a\|_{BMO_t^{-1}} + \|a\|_{BMO_t^{-1}} \\
&\leqslant \sup_{0<\tau<\tau_0} \|u(\tau) - a\|_{BMO_1^{-1}} + \|a\|_{BMO_t^{-1}} \\
&< \frac{\varepsilon}{2c_1} \quad \text{for } 0 < t < T'.
\end{aligned} \tag{3.8}$$

Since we set $\varepsilon < 1/(2c_2)$, it follows that

$$\sup_{0<\tau<\tau_0} \|u(\tau)\|_{BMO_t^{-1}} < \frac{1}{4c_1c_2} \quad \text{for } 0 < t < T'.$$

Then Lemma 3.1 (ii) allows us to construct a mild solution u_τ on $[0, T')$ such that $u_\tau(x,0) = u(x,\tau)$. Particularly, we have

$$\|u_\tau\|_{\mathcal{E}_t} \leqslant c_1 \|u(\tau)\|_{BMO_t^{-1}} + c_2 \|u_\tau\|_{\mathcal{E}_t}^2 \quad \text{for} \quad 0 < t < T'.$$

This implies

$$\|u_\tau\|_{\mathcal{E}_t} \leqslant \frac{1-\sqrt{1-4c_1c_2\|u(\tau)\|_{BMO_t^{-1}}}}{2c_2}. \tag{3.9}$$

Since

$$\frac{1-\sqrt{1-4c_1c_2\|u(\tau)\|_{BMO_t^{-1}}}}{2c_2} \leqslant 2c_1\|u(\tau)\|_{BMO_t^{-1}}, \tag{3.10}$$

it follows from (3.8) that

$$\sup_{0<\tau<\tau_0} \|u_\tau\|_{\mathcal{E}_t} < \varepsilon \quad \text{for} \quad 0<t<T'. \qquad \square$$

Proof of Proposition 3.4. Fix $0<\tau<T/2$. We first show that the mild solution $u(\cdot+\tau)$ is the only mild solution on $[0, T/2)$ with $u(\cdot+\tau)|_{t=0} = u(\tau)$. For this purpose, by Proposition 3.2 we may show

$$\lim_{t\to 0} \|u(\cdot+\tau)\|_{\mathcal{E}_t} = 0. \tag{3.11}$$

We shall estimate the each term of u in the norm of $\mathcal{E}_T$. We have

$$\begin{aligned}\sup_{0<s<t} s^{\frac{1}{2}}\|u(s+\tau)\|_{L^\infty} &\leqslant t^{\frac{1}{2}} \sup_{0<s<t} \|u(s+\tau)\|_{L^\infty} \\ &= t^{\frac{1}{2}} \sup_{\tau<s<t+\tau} \|u(s)\|_{L^\infty}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\sup_{x\in\mathbb{R}^n \; 0<R^2<t} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} \int_0^{R^2} |u(y,s+\tau)|^2\, ds\, dy \right)^{\frac{1}{2}} \\ &\leqslant \; t^{\frac{1}{2}} \sup_{0<s<t} \|u(s+\tau)\|_{L^\infty} \\ &= \; t^{\frac{1}{2}} \sup_{\tau<s<t+\tau} \|u(s)\|_{L^\infty}.\end{aligned}$$

Since $\sup_{\tau<s<t+\tau}\|u(s)\|_{L^\infty}$ is finite, the right-hand sides of the above estimates converge to 0 as t goes to 0, and we obtain (3.11), so we can conclude that the mild solution for the initial value $u(\tau)$ is unique on $[0, T/2)$.

On the other hand, it follows from Lemma 3.5 that for any $\varepsilon > 0$ we choose τ, T' and the mild solution u_τ with the property (3.7) as in Lemma 3.5.

By the uniqueness of $u(\cdot+\tau)$, we have

$$\begin{aligned}\|u(\cdot+\tau)\|_{\mathcal{E}_t} = \|u_\tau\|_{\mathcal{E}_t} < \varepsilon \quad &\text{for } 0<t<\min\{T', T/2\}\\ &\text{and } 0<\tau<\min\{\tau_0, T/2\}.\end{aligned} \tag{3.12}$$

Therefore it suffices to show that

$$\|u\|_{\mathcal{E}_t} \leqslant \lim_{\tau\to 0}\|u(\cdot+\tau)\|_{\mathcal{E}_t}. \tag{3.13}$$

Indeed this estimate and (3.12) imply

$$\|u\|_{\mathcal{E}_t} < \varepsilon \qquad \text{for } 0<t<\min\{T', T/2\}$$

and we obtain the condition (Ad′), since ε is arbitrary.

We show (3.13) by estimating each term of u in the norm of $\mathcal{E}_t$. Both estimates can be obtained by the similar contradiction argument. For the first term:

$$\sup_{0<s<t} s^{\frac{1}{2}}\|u(s)\|_{L^\infty} \leqslant \lim_{\tau\to 0}\sup_{0<s<t} s^{\frac{1}{2}}\|u(s+\tau)\|_{L^\infty}, \tag{3.14}$$

this is equivalent to the following:

$$A \equiv \sup_{0<s<t} s\|u(s)\|_{L^\infty}^2 \leqslant \lim_{\tau\to 0}\sup_{0<s<t} s\|u(s+\tau)\|_{L^\infty}^2 \equiv B. \tag{3.15}$$

Assume that $A > B$, then for $\varepsilon_1 := A - B > 0$ there exists $t_1 \in (0, t)$ such that

$$t_1\|u(t_1)\|_{L^\infty}^2 \geqslant B + \frac{\varepsilon_0}{2}.$$

Let τ be

$$\tau < \min\left\{t_1, \frac{\varepsilon_0}{4\|u(t_1)\|_{L^\infty}^2}\right\}. \tag{3.16}$$

Then we have

$$|(t_1-\tau)\|u(t_1)\|_{L^\infty}^2 - t_1\|u(t_1)\|_{L^\infty}^2| < \tau\|u(t_1)\|_{L^\infty}^2 < \frac{\varepsilon_0}{4}.$$

Hence there exists $t_1' := t_1 - \tau > 0$ such that

$$t_1'\|u(t_1'+\tau)\|_{L^\infty}^2 > t_1\|u(t_1)\|_{L^\infty}^2 - \frac{\varepsilon_0}{4} \geqslant B + \frac{\varepsilon_0}{4}.$$

Since τ is the arbitrary number satisfying (3.16), this contradicts the definition of B.

On the other hand, for the second term on (3.13) it suffices to show

$$\begin{aligned} A' &:= \sup_{x\in\mathbb{R}^n,\ 0<R^2<t} \frac{1}{|B(x,R)|}\int_{B(x,R)}\int_0^{R^2} |u(y,s)|^2\,ds\,dy \\ &\leqslant \lim_{\tau\to 0}\sup_{x\in\mathbb{R}^n,\ 0<R^2<t} \frac{1}{|B(x,R)|}\int_{B(x,R)}\int_0^{R^2} |u(y,s+\tau)|^2\,ds\,dy =: B'. \end{aligned}$$

Assume that $A' > B'$, then for $\varepsilon_2 := A' - B' > 0$ there exists $t_2 \in (0,t)$ such that

$$\sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,t_2^{\frac{1}{2}})|}\int_{B(x,t_2^{\frac{1}{2}})}\int_0^{t_2} |u(y,s)|^2\,ds\,dy \geqslant B' + \frac{\varepsilon_2}{2}.$$

By the absolutely continuity of the integral, there exists $\tau_2 > 0$ such that

$$\begin{aligned} &\left|\sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,t_2^{\frac{1}{2}})|}\int_{B(x,t_2^{\frac{1}{2}})}\int_0^{t_2} |u(y,s)|^2\,ds\,dy\right. \\ &\qquad \left. - \sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,t_2^{\frac{1}{2}})|}\int_{B(x,t_2^{\frac{1}{2}})}\int_\tau^{t_2+\tau} |u(y,s)|^2 ds dy\right| < \frac{\varepsilon_2}{4} \end{aligned}$$

for all $\tau < \tau_2$.

Hence we have

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,t_2^{\frac{1}{2}})|} \int_{B(x,t_2^{\frac{1}{2}})} \int_0^{t_2} |u(y,s+\tau)|^2\,ds\,dy \\
&\quad = \sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,t_2^{\frac{1}{2}})|} \int_{B(x,t_2^{\frac{1}{2}})} \int_\tau^{t_2+\tau} |u(y,s)|^2\,ds\,dy \\
&\quad > \sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,t_2^{\frac{1}{2}})|} \int_{B(x,t_2^{\frac{1}{2}})} \int_0^{t_2} |u(y,s)|^2\,ds\,dy - \frac{\varepsilon_2}{4} \geqslant B' + \frac{\varepsilon_2}{4}.
\end{aligned}
$$

This contradicts the definition of B'. Thus we can complete the proof. □

3.2. *Proof of Theorem 2.4*

We notice that we may assume that there exists a mild solution u in $\mathcal{E}_T$ by Lemma 3.1. So it suffices to show that u belongs to $C([0,T);gmo^{-1})$. For that purpose we divide the proof into 3 steps.

Step 1: Firstly we show that $u(t)$ is uniformly bounded in bmo^{-1} on $[0,T)$. We may show the following estimates:

$$
\sup_{0<t<T} \|e^{t\Delta}a\|_{bmo^{-1}} \leqslant c\|a\|_{bmo^{-1}}, \tag{3.17}
$$

$$
\sup_{0<t<T} \|V\mathrm{P}\nabla\cdot(u\otimes u)(t)\|_{bmo^{-1}} \leqslant c\|u\|_{\mathcal{E}_T}^2, \tag{3.18}
$$

where we denote $Vf(t) := \int_0^t e^{(t-s)\Delta} f(s)\,ds$. The former is obtained by Minkowski's integral inequality. For the latter, it follows that

$$
\begin{aligned}
\|V\mathrm{P}\nabla\cdot(u\otimes u)(t)\|_{bmo^{-1}} &\leqslant \|V\mathrm{P}\nabla\cdot(u\otimes u)(t)\|_{BMO^{-1}} \\
&= \|V\mathrm{P}(u\otimes u)(t)\|_{BMO} \\
&\leqslant c\|V(u\otimes u)(t)\|_{L^\infty},
\end{aligned}
$$

where we use the Carleson characterization of *BMO* norm and the boundedness of P in *BMO* [16, Chapter 4]. Since the estimate of (3.18) is translation invariant to the space-variable, we may show

$$|V(u \otimes u)(0,t)| \leqslant c\|u\|_{\mathcal{E}_T}^2 \quad \text{for} \quad 0 < t < T.$$

By the estimate of the heat kernel:

$$G_t(x) \leqslant ct^{\frac{1}{2}}(t^{\frac{1}{2}} + |x|)^{-n-1}, \tag{3.19}$$

we have

$$\begin{aligned}
&|V(u \otimes u)(0,t)| \\
&\leqslant ct^{\frac{1}{2}} \left(\int_{B(0,2t^{\frac{1}{2}})} \int_0^t \frac{|u(y,s)|^2}{((t-s)^{\frac{1}{2}} + |y|)^{n+1}}\, ds\, dy \right. \\
&\quad \left. + \int_{B(0,2t^{\frac{1}{2}})^c} \int_0^t \frac{|u(y,s)|^2}{((t-s)^{\frac{1}{2}} + |y|)^{n+1}}\, ds\, dy \right) \\
&=: ct^{\frac{1}{2}}\,(I + II)\,.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
I &\leqslant \int_{B(0,2t^{\frac{1}{2}})} \int_0^{\frac{t}{2}} \frac{|u(y,s)|^2}{((t-s)^{\frac{1}{2}} + |y|)^{n+1}}\, ds\, dy + \int_{B(0,2t^{\frac{1}{2}})} \int_{\frac{t}{2}}^t \frac{|u(y,s)|^2}{((t-s)^{\frac{1}{2}} + |y|)^{n+1}}\, ds\, dy \\
&\leqslant c \int_{B(0,2t^{\frac{1}{2}})} \int_0^{\frac{t}{2}} \frac{|u(y,s)|^2}{t^{\frac{n+1}{2}}}\, ds\, dy + c \sup_{\frac{t}{2} < s < t} s\|u(s)\|_{L^\infty}^2 \\
&\quad \times \int_{B(0,2t^{\frac{1}{2}})} \int_{\frac{t}{2}}^t s^{-1}((t-s)^{\frac{1}{2}} + |y|)^{-n-1}\, ds\, dy \\
&\leqslant ct^{-\frac{1}{2}} \sup_{x \in \mathbb{R}^n} |B(x, t^{\frac{1}{2}})|^{-1} \int_{B(x,t^{\frac{1}{2}})} \int_0^t |u(y,s)|^2\, ds\, dy + ct^{-\frac{1}{2}} \sup_{\frac{t}{2} < s < t} s\|u(s)\|_{L^\infty}^2 \\
&\leqslant ct^{-\frac{1}{2}} \|u\|_{\mathcal{E}_T}^2\,.
\end{aligned}$$

On the other hand, it follows that

$$II \leqslant \int_{B(0,2t^{\frac{1}{2}})^c} \int_0^t \frac{|u(y,s)|^2}{|y|^{n+1}}\, ds\, dy.$$

We cover $B(0, 2t^{\frac{1}{2}})^c$ by a family of balls centered at $t^{\frac{1}{2}}\mathbb{Z}^n$ $(= \{(t^{\frac{1}{2}}x_i)_{i=1,\dots,n};\, x_i \in \mathbb{Z}^n\})$ with radius $t^{\frac{1}{2}}$. Furthermore classify the center by the $Q(0, t^{\frac{1}{2}}m)$, where $Q(0, t^{\frac{1}{2}}m)$ is the cube with the center at 0 with side length $2t^{\frac{1}{2}}m$. We have

$$II \leqslant c \sum_{m=2}^{\infty} \sum_{x \in Q(0,t^{\frac{1}{2}}m) \cap t^{\frac{1}{2}}\mathbb{Z}^n} \int_{B(x,t^{\frac{1}{2}})} \int_0^t (t^{\frac{1}{2}}m)^{-n-1} |u(y,s)|^2\, ds\, dy.$$

Since the number of the lattice point on the cube is proportional to the measure of the surface, there holds

$$\begin{aligned} II &\leqslant c\, t^{-\frac{n+1}{2}} \sum_{m=2}^{\infty} \frac{1}{m^2} \sup_{x \in \mathbb{R}^n} \int_{B(x,t^{\frac{1}{2}})} \int_0^t |u(y,s)|^2\, ds\, dy \\ &\leqslant c\, t^{-\frac{1}{2}} \sup_{x \in \mathbb{R}^n} |B(x, t^{\frac{1}{2}})|^{-1} \int_{B(x,t^{\frac{1}{2}})} \int_0^t |u(y,s)|^2\, ds\, dy. \\ &\leqslant c\, t^{-\frac{1}{2}} \|u\|_{\mathcal{E}_T}^2. \end{aligned}$$

Hence we have

$$I + II \leqslant c t^{-\frac{1}{2}} \|u\|_{\mathcal{E}_T}^2. \tag{3.20}$$

This implies (3.18).

Step 2: Next we verify that $u(t_0)$ belongs to gmo^{-1} for $t_0 \in [0, T)$. Obviously, $u(t_0)$ belongs to gmo^{-1} at $t_0 = 0$. So we may assume $t_0 > 0$, then there holds that

$$\begin{aligned} &\|e^{t\Delta} u(t_0) - u(t_0)\|_{bmo^{-1}} \\ &\quad \leqslant \|e^{(t+t_0)\Delta} a - e^{t_0\Delta} a\|_{bmo^{-1}} + \|e^{t\Delta} B(u,u)(t_0) - B(u,u)(t_0)\|_{bmo^{-1}} \\ &\quad \leqslant \|e^{t\Delta} a - a\|_{bmo^{-1}} + \|e^{t\Delta} B(u,u)(t_0) - B(u,u)(t_0)\|_{bmo^{-1}}. \end{aligned}$$

By the definition of gmo^{-1}, the linear term vanishes as t goes to 0 . For the nonlinear term, we have

$$
\begin{aligned}
&\|e^{t\Delta}B(u,u)(t_0) - B(u,u)(t_0)\|_{bmo^{-1}} \\
&\quad = \left\| \int_0^{t_0} (e^{(t_0+t-s)\Delta} - e^{(t_0-s)\Delta})\mathrm{P}\nabla \cdot (u \otimes u)(s)\, ds \right\|_{bmo^{-1}} \\
&\quad \leqslant \left\| \int_0^{t_0} (e^{(t_0+t-s)\Delta} - e^{(t_0-s)\Delta})\mathrm{P}\nabla \cdot (u \otimes u)(s)\, ds \right\|_{BMO^{-1}} \\
&\quad \leqslant c \left\| \int_0^{t_0} (e^{(t_0+t-s)\Delta} - e^{(t_0-s)\Delta})\mathrm{P}(u \otimes u)(s)\, ds \right\|_{BMO} \\
&\quad \leqslant c \left\| \int_0^{t_0} (e^{(t_0+t-s)\Delta} - e^{(t_0-s)\Delta})(u \otimes u)(s)\, ds \right\|_{L^\infty}.
\end{aligned}
$$

By translating to space-variable, it suffices to show

$$
\left| \int_0^{t_0} (e^{(t_0+t-s)\Delta} - e^{(t_0-s)\Delta})(u \otimes u)(0,s)\, ds \right| \leqslant c t^{\frac{1}{2}} t_0^{-\frac{1}{2}} \|u\|_{\mathcal{E}_T}^2. \tag{3.21}
$$

In order to prove this estimate, we use the following lemma.

Lemma 3.6 (*Mean value theorem*). *Let* $a > b > 0$. *Assume that* f *is a continuous function on* $[a,b]$ *and set* $g(t) := f(t^2)$. *Then for* $t_0 \in [a,b]$ *and* $t \in [0, b-t_0]$, *it follows that*:

$$
|f(t_0+t) - f(t_0)| \leqslant t^{\frac{1}{2}} \sup_{\theta \in [0,1]} g'((t_0+t\theta)^{\frac{1}{2}}).
$$

The proof of this lemma is easy, so we may omit it.

Let $f(t) = G_{t-s}$, then the estimate of the heat kernel yields that

$$
|G_{t_0+t-s}(y) - G_{t_0-s}(y)| \leqslant c t^{\frac{1}{2}} \sup_{\theta \in [0,1]} ((t_0+t\theta-s)^{\frac{1}{2}} + |y|)^{-n-1}.
$$

Hence we have

$$\begin{aligned}
&\left| \int_0^{t_0} (e^{(t_0+t-s)\Delta} - e^{(t_0-s)\Delta})(u \otimes u)(0,s)\, ds \right| \\
&\quad \leqslant c t^{\frac{1}{2}} \left(\int_{B(0,2t_0^{\frac{1}{2}})} \int_0^{t_0} \frac{|u(y,s)|^2}{((t_0-s)^{\frac{1}{2}} + |y|)^{n+1}}\, ds\, dy \right. \\
&\qquad \left. + \int_{B(0,2t_0^{\frac{1}{2}})^c} \int_0^{t_0} \frac{|u(y,s)|^2}{((t_0-s)^{\frac{1}{2}} + |y|)^{n+1}}\, ds\, dy \right) \\
&\quad =: c t^{\frac{1}{2}} (I + II).
\end{aligned}$$

The right-hand side can be handled in the same way as (3.20), so we obtain (3.21).

Step 3: Finally we shall show the right-continuity of u in time. The left-continuity is obtained in the similar way, so we omit it. Let $t_0 \in (0,T)$ and $t > 0$, we have

$$\begin{aligned}
&\|u(t_0+t) - u(t_0)\|_{bmo^{-1}} \\
&\quad \leqslant \|(e^{(t_0+t)\Delta} - e^{t_0\Delta})a\|_{bmo^{-1}} + \|B(u,u)(t_0+t) - B(u,u)(t_0)\|_{bmo^{-1}} \\
&\quad \leqslant \|e^{t\Delta}a - a\|_{bmo^{-1}} + \left\| \int_0^{t_0} (e^{(t_0+t-s)\Delta} - e^{(t_0-s)\Delta}) \mathrm{P}\nabla \cdot (u \otimes u)(s) ds \right\|_{bmo^{-1}} \\
&\qquad + \left\| \int_{t_0}^{t_0+t} e^{(t_0+t-s)\Delta} \mathrm{P}\nabla \cdot (u \otimes u)(s)\, ds \right\|_{bmo^{-1}} \\
&\quad \leqslant \|e^{t\Delta}a - a\|_{bmo^{-1}} + \left\| \int_0^{t_0} (e^{(t_0+t-s)\Delta} - e^{(t_0-s)\Delta}) \mathrm{P}\nabla \cdot (u \otimes u)(s)\, ds \right\|_{bmo^{-1}} \\
&\qquad + \left\| \int_{t_0}^{t_0+t} e^{(t_0+ts)} (u \otimes u)(s)\, ds \right\|_{L^\infty}.
\end{aligned}$$

The first two terms are estimated in *Step* 2. For the last term, by translating to space-variable, it suffices to show the case $x = 0$. By the estimate of heat kernel (3.19), we have

$$\begin{aligned}
&\left| \int_{t_0}^{t_0+t} e^{(t_0+t-s)\Delta}(u\otimes u)(0,s)\,ds \right| \\
&\leqslant c(t_0+t)^{\frac{1}{2}} \int_{\mathbb{R}^n} \int_{t_0}^{t_0+t} \frac{|u(y,s)|^2}{((t_0+t-s)^{\frac{1}{2}}+|y|)^{n+1}}\,ds\,dy \\
&\leqslant c(t_0+t)^{\frac{1}{2}} \sup_{0<s<t_0+t} s\|u(s)\|_{L^\infty}^2 \int_{\mathbb{R}^n} \int_{t_0}^{t_0+t} s^{-1}((t_0+t-s)^{\frac{1}{2}}+|y|)^{-n-1}\,ds\,dy \\
&\leqslant ct^{\frac{1}{2}} t_0^{-\frac{1}{2}} \|u\|_{\mathcal{E}_{t_0+t}}^2,
\end{aligned}$$

which tends to 0 as t goes to 0.

In the case $t_0 = 0$, it follows from (3.18) that

$$\begin{aligned}
\|u(t)-u(0)\|_{bmo^{-1}} &\leqslant \|e^{t\Delta}a - a\|_{bmo^{-1}} + \|V\mathrm{P}\nabla(u\otimes u)(t)\|_{bmo^{-1}} \\
&\leqslant \|e^{t\Delta}a - a\|_{bmo^{-1}} + c\|u\|_{\mathcal{E}_t}^2.
\end{aligned}$$

Recalling the inequality (3.9) and (3.10) ($\tau = 0$), we obtain

$$\|u\|_{\mathcal{E}_t} \leqslant \|a\|_{BMO_t^{-1}}.$$

Since $a \in vmo^{-1}$, the right-hand side vanishes as t goes to 0. Thus the proof is complete. □

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